For $\nu(\mathbf{x}) = \nu \equiv \mathrm{const}$ the functions v_{ik} are evaluated explicitly by using the three-dimensional Fourier integral transform

$$v_{ik}(\mathbf{x}, \boldsymbol{\xi}) = \frac{2\pi}{1-\nu} \frac{\partial^2 |\mathbf{x}-\boldsymbol{\xi}|}{\partial x_i \partial x_k}$$

We hence obtain the known representation of the fundamental solution of the equilibrium equation of homogeneous elasticity theory, the Kelvin matrix /6/.

We note that by using relationships (1.7), (3.3) and (3.4), the stress field caused by the action of a unit concentrated force on an unbounded elastic body with inhomogeneity of the type consideration can be evaluated.

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Translated by M.D.F.

PMM U.S.S.R., Vol.51, No.4, pp.479-488, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON CERTAIN FORMULATIONS OF THE BOUNDARY-ELEMENTS METHOD*

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Variational formulations are proposed for the boundary-element method (BEM) to solve linear problems of elasticity theory with a known Green's tensor. Unlike existing BEM formulations utilizing the method of weighted residuals /1/ or boundary integral equations /2/, the formulations to be considered below use a variational formulation of the problems for boundary functionals /3/ in a set of allowable functions in the form of double-layer potentials whose density is given in the form of BEM basis functions. Also examined is a BEM variational formulation on the basis of minimization problems for Trefftz generalized functionals of the fundamental boundary value problems of linear elasticity theory /4/. A basis for the formulations is presented. Utilization of the proposed BEM formulations is effective in solving boundary-contact problems; consequently, a numerical realization is examined with an example of a unilateral variational problem (of the generalized Signorini problem type /5/) corresponding to the classical contact problem of inserting a stamp

**Prikl.Matem.Mekhan.*, 51, 4, 616-627, 1987

into an elastic half-plane with approximation of the possible contact boundary by isoparametric boundary elements.

1. Let us consider variational formulations of problems for boundary functionals (BF) utilized in the BEM variational formulations. Dual variational problems for BF of linear elasticity theory are formulated in /3/ in terms of surface displacements and stresses to whose solution by the method of orthogonal expansions on the domain boundary /6/ the solution of the fundamental boundary value problems of linear elasticity theory is reduced.

The functionals of these problems are minimized (maximized) by solutions of a homogeneous equilibrium equation for an elastic medium in displacements, which it is natural to consider as a constraint of the variational problems. Thus, according to /3/, construction of the solution of the first fundamental problem (the problem with a clamped boundary) reduces to finding the vector $\varphi_0(x)$, $x \in \bar{G}$, which is a solution of the variational problem for the BF

$$F_S(\varphi) = \int_S \varphi \mathbf{t}^{(\nu)}(\varphi)\, ds - 2\int_S \varphi \mathbf{t}^{(\nu)}(\mathbf{u}^*)\, ds \tag{1.1}$$

defined in the subspace $W_2^{*1/2}(S) \subset W_2^{1/2}(S)$ of traces of the solutions of the equilibrium equation $\mathbf{A}\varphi = 0$ in the domain of an elastic medium G where the vector $\mathbf{u}^*(x)$, $x \in \bar{G}$, which is assumed to be given, should satisfy the equilibrium equation $\mathbf{A}\mathbf{u}^* = \mathbf{K}$ ($\mathbf{K}$ is the mass force vector) and the boundary condition on the free part of the boundary (since the whole boundary is fixed in the first problem, there is no such condition). Then if the vector φ_0 is constructed, we obtain /6/ a solution of the first problem $\mathbf{u}_0 = \mathbf{u}^* - \varphi_0$.

According to /5/, and also by using an auxiliary mixed boundary value problem for the vector $\mathbf{u}^*$ with zero boundary condition at points of the boundary of possible contact S_1, unilateral boundary value problems (of the generalized Signorini problem type /7/), reduce to a unilateral variational problem for the BF

$$F_{S_1}(\varphi) = \frac{1}{2}\int_{S_1} \varphi \mathbf{t}^{(\nu)}(\varphi)\, ds + \int_{S_1} \varphi \mathbf{t}^{(\nu)}(\mathbf{u}^*)\, ds \tag{1.2}$$

defined on a convex closed set of vector-functions

$$V(S_1) = \{\varphi \in W_2^{*1/2}(S_1) \mid \varphi^{(\nu)}|_{S_1} \geqslant 0\}$$

where $W_2^{*1/2}(S_1)$ is a subspace of traces of the vector-function φ satisfying the conditions /5/

$$\mathbf{A}\varphi = 0 \quad \text{in} \quad G, \quad \mathbf{t}^{(\nu)}(\varphi)|_{S_2} = 0, \quad S_2 = S|_{S_1} \tag{1.3}$$

A variational formulation of the BEM is possible on the basis of a generalized Trefftz method since minimization of the generalized Trefftz functionals (GTF) reduces to solving equations in the boundary values of the desired solution. For instance, for the second fundamental problem of linear elasticity theory (the problem with a free boundary) by analogy with the GTF of the Neumann boundary value problem for Poisson's equation /8/ the GTF can be taken in the form

$$\Phi(\mathbf{u}) = 2\int_G W(\mathbf{u})\, dG + \frac{1}{\alpha}\int_S (\mathbf{t}^{(\nu)}(\mathbf{u}) - \alpha\mathbf{u})^2\, ds - \alpha\int_S \mathbf{u}^2\, ds \tag{1.4}$$

Here $2W(\mathbf{u})$ is the quadratic form of linear elasticity theory /8/, $\mathbf{t}^{(\nu)}(\mathbf{u})$ is the surface normal stress vector, and α is a certain positive constant whose value is selected from the condition for the functional $\Phi(\mathbf{u})$ to be bounded from below. A generalization of the functional (1.4) was presented /4/ for the case when norms of the boundary displacement and stress values in the classes of functions $W_2^{1/2}(S)$ and $W_2^{-1/2}(S)$, respectively, are inserted into the functions, resulting in refinement of the values of the functional in the solution of the problem as compared with a functional of the form (1.4).

The GTF of the fundamental boundary value problems of linear elasticity theory are minimized in solutions of the equilibrium equation in the displacement $\mathbf{A}\mathbf{u} = \mathbf{K}$. In problems with a free boundary the allowable functions of the variational problems should be subjected to known conditions ensuring the single-valued solvability of the problems that are presented below in the BEM formulations.

2. Let $G \subset E_m$ ($m = 2, 3$) be the domain of an elastic medium with a sufficiently smooth boundary S. We partition the boundary S into boundary elements according to /2/: let S_Δ, the boundary approximating S, consist of the boundary elements Δs_n, that is

$$S_\Delta = \bigcup_{n=1}^{N} \Delta s_n \tag{2.1}$$

and let $G_\Delta \subset G$ be the domain bounded by S_Δ; we will assume that $G_\Delta \to G$ as $\operatorname{diam} \Delta s_n \to 0$. Let the displacement field at the points $y \in \Delta s_n$ be interpolated at the nodal values of the displacements by using the vector functions

$$\mathbf{w}_n(y) = [\mathrm{W}]\,\Psi, \quad n = 1, \ldots, N$$

where [W] is the matrix of the nodal displacements of the element Δs_n, and Ψ is a vector of the basis functions, i.e., each component of the displacements of the points $\eta \in \Delta s_n$ has the form

$$w_{in}(y(\eta)) = \sum_{k=1}^{K} W_{ik}\psi_K(\eta), \quad i = 1, \ldots, m \tag{2.2}$$

Here $k = 1, \ldots, K$ is the number of nodes of the element Δs_n; W_{ik} are elements of the nodal displacement matrix, and $\psi_k(\eta)$ is the basis function corresponding to the node k, where $\{\eta\}$ is the local coordinate system of points of the boundary element Δs_n which is planar or curvilinear depending on the kind of boundary elements. The connection of the coordinate system $\{\eta\}$ with the global (Cartesian) coordinate system $\{y\}$ for points of the element Δs_n is set up /2/ in terms of the Jacobi matrix $[J]$ of the transformation that is defined by the equation $y = y(\eta)$, $\eta \in \Delta s_n$, so that $dy = [J]\,d\eta$.

Completeness in $L_2(S_\Delta)$ is assumed for the sequence of interpolation functions $\{w_{in}\}_{n=1,2,\ldots}$ of the form (2.2), which is ensured /2/ by the representation of $w_{in}(\eta)$ on a finite size element Δs_n in the form of a complete polynomial of the variable $\eta \in \Delta s_n$ and is understood in the sense of convergence of the linear combination $\alpha_1 w_{i1} + \alpha_2 w_{i2} + \ldots + \alpha_N w_{iN}$ in the mean to a certain function w_i $(i = 1, \ldots, m)$ as $N \to \infty$ $(\operatorname{diam} \Delta s_n \to 0)$ because of the selection of the constants α_n.

Furthermore, we will consider the case of a homogeneous isotropic elastic medium. Using Green's tensor of the first fundamental problem of statics $\Gamma_{(1)}(x, y)$ (sufficiently complete conditions for the existence of such a tensor are presented in /9/) we construct the sequence of functions

$$\{\beta_{in}\}_{n=1,2,\ldots}, \quad i = 1, \ldots, m \tag{2.3}$$

in the form of double-layer potentials (the notation is borrowed from /9/)

$$\beta_{in}(x) = -\frac{1}{2}\int_{S_\Delta} [\mathbf{T}(\partial/\partial y, \mathbf{v}(y))\,\Gamma_{(1)}(x, y)]^* w_{in}(y)\,ds(y), \quad x \in G_\Delta \tag{2.4}$$

where $\mathbf{T}$ is the boundary vector operator of differentiation with respect to the direction of the positive normal $\mathbf{v}$ to the surface (2.1). The density of the potentials (2.4) is determined in terms of the displacement components of the form (2.2) for points of the elements Δs_n, and therefore $\beta_{in}(x)$ are essentially the displacement components of the points $x \in G_\Delta$ of an elastic medium.

We will investigate the integral in (2.4). Firstly, the integral representation (2.4) is given a basis /9/ for the case of the piecewise-smooth boundary (2.1). Secondly, since the support of the basis functions $\psi_k(\eta)$ is the boundary element Δs_n (as usual, the support is understood to be the set of points $\eta \in S_\Delta$ in which $\psi_k \neq 0$), the integral in (2.4) actually extends only over the cell Δs_n (for each n) and exists /9/ for a continuous displacement field at points of the element Δs_n, which is satisfied if the interpolation functions $w_{in}(\eta)$ are represented in the form of first (and higher) degree polynomials in the variable $\eta \in \Delta s_n$.

The functions $\beta_{in}(x)$, $x \in G_\Delta$, defined by (2.4) have all the properties inherent in a double-layer surface potential, in particular, $\forall x \in G_\Delta$ are defined and finite and are /9/ solutions (for every n) of the first problem of statics in the domain G_Δ with boundary S_Δ

$$\mathbf{A}\beta_{in}(x) = 0, \quad x \in G_\Delta, \quad \beta_{in}|_{S_\Delta} = w_{in}(y), \quad i = 1, \ldots, m \tag{2.5}$$

$$\mathbf{A} \equiv \mu\Delta + (\lambda + \mu)\,\mathrm{grad}\,\mathrm{div}$$

($\mathbf{A}$ is the isotropic elasticity theory operator). The foundation of the formulation of problem (2.5) for the case of the piecewise-smooth boundary (2.1) is a consequence of the foundation of the formula (2.4). By proposing to utilize the functions $\beta_{in}(x)$ later as basis functions of the Ritz process in the variational formulation of the BEM, using the BF (1.1) and (1.2), it is necessary to establish the basis properties of the sequence (2.3).

Because of the linearity of the integral transformation (2.4), the functions β_{in} form a sequence of linearly independent functions. The property of completeness of the sequence (2.3) in $L_2(S_\Delta)$ follows from the euality $\beta_{in}|S_\Delta = w_{in}$ (see (2.5)) and the assumed completeness of the interpolation functions $\{w_{in}\}_{n=1,2,\ldots}$ in $L_2(S_\Delta)$. A boundary value problem with given stresses on the boundary S_Δ: $\mathbf{t}^{(\mathbf{v})}(\boldsymbol{\varphi})|_{S_\Delta} = \mathbf{t}^{(\mathbf{v})''}(\mathbf{u}^*)$ corresponds /3/ to the variational problem for the BF (1.1) with integrals over the boundary S_Δ and with the constraint $\mathbf{A}\boldsymbol{\varphi} = 0$ in G_Δ. As is well-known /8/, when solving such a problem by an energy method, the application of the

Ritz process assumes completeness of the sequence of basis functions "in energy" for the operator of the boundary value problem. This completeness of the sequence (2.3) will hold (see /8/, p.360) by virtue of the positive-definiteness of the operator of the second fundamental problem (in the domain G_Δ with boundary S_Δ) under the condition of completeness of the sequence (2.3) in the space $L_2(G_\Delta)$.

This latter is established as follows. Let the estimate

$$c_0^2 = \iint\limits_{G_\Delta S_\Delta} [\mathbf{T}(\partial/\partial y, \mathbf{v}(y))\,\mathbf{\Gamma}_{(1)}(x, y)]^{*2}\, ds(y)\, dG_\Delta(x) < \infty$$

hold, which means essentially $[\mathbf{T}(\partial/\partial y, \mathbf{v}(y))\,\mathbf{\Gamma}_{(1)}(x, y)]^* \in L_2(G_\Delta) \times L_2(S_\Delta)$ and is satisfied /10/ in potential-theory problems. We square both sides of the equality (2.4) and apply the Cauchy inequality to the right side. Then integrating both sides of this inequality with respect to G_Δ, we obtain the inequality

$$\|\beta_{in}\|^2_{L_2(G_\Delta)} \leqslant {}^1/_4\, c_0^2 \|w_{in}\|^2_{L_2(S_\Delta)}, \quad i = 1, \ldots, m$$

on whose basis the completeness of the sequence (2.3) in $L_2(S_\Delta)$ follows from the completeness of the sequence of functions $\{w_{in}\}_{n=1,2,\ldots}$ in $L_2(G_\Delta)$.

3. We proceed to a BEM formulation on the basis of the variational problem for the BF (1.1) with a constraint on the allowable vector-functions $\mathbf{A}\boldsymbol{\varphi}(x) = 0,\ x \in G$. The boundary-element approximation of this problem uses partitioning of the boundary S in the form (2.1) according to Sect.2 and approximation of the solution "according to Ritz" by basis functions in the form of the potentials (2.4).

Let Φ_{ik} be the components of the desired nodal displacements; then the i-th component of the approximate solution will have the form

$$\varphi_{iN} = \sum_{n=1}^{N}\sum_{k=1}^{K} \Phi_{ik}\beta_{nk}(x), \quad x \in G_\Delta, \quad i = 1, \ldots, m \tag{3.1}$$

where the functions β_{nk} defined according to (2.4) when taking account of the representation of the interpolation functions $\varphi_{in}(y(\eta)),\ \eta \in \Delta S_n$ in the form (2.2) will be determined from the formula

$$\beta_{nk}(x) = -\frac{1}{2}\int\limits_{\Delta S_n}\Big[\mathbf{T}\Big(\frac{\partial}{\partial y(\eta)}, \mathbf{v}_n(y(\eta))\Big)\mathbf{\Gamma}_{(1)}(x, y(\eta))\Big]^* \times \tag{3.2}$$
$$\psi_k(\eta)\,|J|\, ds_n(\eta) \quad n = 1, \ldots, N,\ k = 1, \ldots, K$$

($|J|$ is the determinant of the Jacobi matrix $[J]$ that transforms $ds(\eta)$ into $ds(y)$).

A Ritz system of linear algebraic equations to find the nodal values Φ_{ik} follows from the condition for the minimum of the functional $F_{S_\Delta}(\varphi_{iN})$

$$\sum_{n=1}^{N}\Big[\int\limits_{\Delta s_n(y)} \mathbf{t}^{(\nu_n)}\Big(\sum_{i=1}^{m}\sum_{k=1}^{K}\Phi_{ik}\beta_{nk}\Big)\beta_{rp}\, ds_n(y) - \tag{3.3}$$
$$\int\limits_{\Delta s_n(y)} \mathbf{t}^{(\nu_n)}(\mathbf{u}_n^*)\beta_{rp}\, ds_n(y)\Big] = 0 \quad r = 1, \ldots, N,\ p = 1, \ldots, K$$

($\mathbf{v}_n$ and $\mathbf{u}_n^*$ are the positive normal and the value of the given vector $\mathbf{u}^*$ at the points $y \in \Delta s_n$). The solvability condition for the boundary-element approximation of the variational problem for the functional $F_{S_\Delta}(\varphi_{iN})$ is written in the form ($\mathbf{r}_n$ is the radius-vector of the point $y \in \Delta s_n$)

$$\sum_{n=1}^{N}\int\limits_{\Delta S_n} \mathbf{t}^{(\nu_n)}(\mathbf{u}_n^*)\, dS_n = \sum_{n=1}^{N}\int\limits_{\Delta s_n} \mathbf{t}^{(\nu_n)}(\mathbf{u}_n^*) \times \mathbf{r}_n\, ds_n = 0$$

We transform system (3.3) to the final system of BEM equations.

To do this, it should be taken into account that the boundary values of the functions $\beta_{nk}(x)$ $(x \in G_\Delta)$, defined according to (3.2) are the following

$$\beta_{nk}(\eta)|_{\Delta s_n} = \psi_k(\eta), \quad \eta \in \Delta s_n, \quad k = 1, \ldots, K \tag{3.4}$$

We obtain for the operator of the boundary stresses at points of the element Δs_n

$$\mathbf{T}\Big(\frac{\partial}{\partial y(\eta)}, \mathbf{v}_n(y(\eta))\Big)\Big(\sum_{i=1}^{m}\sum_{k=1}^{K}\Phi_{ik}\beta_{nk}(\eta)\Big) \equiv \tag{3.5}$$
$$\mathbf{t}^{(\nu_n)}\Big(\sum_{i=1}^{m}\sum_{k=1}^{K}\Phi_{ik}\beta_{nk}\Big) = \sum_{i=1}^{m}\sum_{k=1}^{K}\Phi_{ik}\, c(\lambda, \mu)\frac{\partial\beta_{nk}}{\partial\mathbf{v}_n}$$

where $c(\lambda, \mu)$ is a constant factor dependent on the Lamé constants.

Taking account of Eq.(3.4), the result of differentiating it with respect to $\nu_n(\eta)$, and expression (3.5), we reduce system (3.3) in the unknowns Φ_{ik} to a system of BEM equations

$$\sum_{n=1}^{N}\sum_{k=1}^{K}\Phi_{ik}\int_{\Delta s_n(\eta)} c(\lambda,\mu)\frac{\partial\psi_k}{\partial\nu_n}\psi_p|J|\,ds_n(\eta)= \sum_{n=1}^{N}\sum_{\rho=1}^{K}\int_{\Delta S_n(\eta)} \mathbf{t}^{(\nu_n)}(\mathbf{u}_n^*)\psi_p|J|\,ds_n(\eta),\quad i=1,\ldots,m \tag{3.6}$$

The vector φ_N of the approximate "Ritz" solution of the boundary-element approximation of the variational problem for the functional $F_{S_\Delta}(\varphi_N)$ should be subjected to the single-valuedness condition

$$\int_{G_\Delta}\varphi_N\,dG_\Delta=\int_{G_\Delta}\operatorname{rot}\varphi_N\,dG_\Delta=0 \tag{3.7}$$

The matrix of system (3.6) whose elements are evaluated in terms of integrals of the form

$$\int_{\Delta s_n(\eta)}\sum_{k=1}^{K}\frac{\partial\psi_k}{\partial\nu_n}\psi_p|J|\,ds_n(\eta),\quad p=1,\ldots,K$$

is symmetric (as is usual for a variational formulation of the problem). The positive-definiteness of the matrix follows from the positive-definiteness of the quadratic form $2\int_{G_\Delta}W(\varphi_N)\,dG_\Delta$ for the vectors φ_N satisfying condition (3.7). The single-valued solvability of the system of BEM Eqs.(3.6) follows from the symmetry and positive-definiteness of the matrix.

It should be taken into account for the foundation of the BEM formulation described that the variational problem for the BF (1.1) with a constraint on the allowable functions $A\varphi=0$ in G is equivalent to the second fundamental problem in the variational formulation for the energy functional, and the boundary-element approximations of the form (3.1) are "Ritz" approximations of the solution of this problem in the domain G_Δ with boundary S_Δ. Indeed, it is sufficient to confirm compliance with the conditions to which the basis functions β_{in} of the boundary-element approximations φ_{iN} in the Ritz process (/8/, p.96) are subject.

1°. For any N, the elements $\beta_{i1},\beta_{i2},\ldots,\beta_{iN}$ are linearly independent.

2°. The elements β_{in} should belong to the energy space of the second fundamental problem of linear elasticity theory formulated in the domain G_Δ with the boundary S_Δ, which, as is well-known /8/, is a subspace of the Sobolev class of vector-functions $W_2^1(G_\Delta)$ that satisfy condition (3.7). The elements $\beta_{in}(x)\ (x\in G_\Delta)$, defined according to (2.4) are at least twice continuously differentiable in the domain G_Δ with piecewise-smooth boundary S_Δ (by virtue of (2.5)); such a set of functions is included continuously in the space $W_2^1(G_\Delta)$ by virtue of the imbedding theorem /8/.

3°. The sequence (2.3) is the complete, "in energy", operator of the second fundamental problem of linear elasticity theory, as is proved in Sect.2.

On the basis of the above, approximations of the form (3.1), where the coefficients Φ_{ik} are determined from the system of BEM Eqs.(3.6) under conditions (3.7), form a minimizing sequence of the Ritz process for the problem of minimizing the energy functional of the second boundary value problem in the domain G_Δ with the boundary S_Δ. The question of the convergence of the minimizing sequence $\{\varphi_{iN}\}$ as the size of the boundary element $(\operatorname{diam}\Delta s_n\to 0\Rightarrow N\to\infty)$ diminishes to the exact (energy) solution φ_0 of the second problem should be considered /2, 11/ upon satisfying certain conditions on the interpolation functions $\varphi_{in}\ (i=1,\ldots,m)$ of the form (2.2) of the boundary element $\Delta s_n\subset S_\Delta$.

The convergence of the boundary-element approximation as the size of the boundary element diminishes is ensured /2, 11/ by the condition of completeness of the basis functions of an element and the condition of consistency of the elements, which is that the interpolation function and its derivatives to order $q-1$ inclusive should be continuous during passage through the boundary elements, where q is the highest order of the derivatives contained in the functional. Completeness criteria for the basis functions of a finite element and the consistency of the finite elements ensuring convergence of the FEM (finite element method) in the "Ritz" formulation method are presented in /11/. Since the boundary-element approximation considered above for the variational problems for a BF of the form (1.1) is essentially a two-dimensional finite-element approximation, the above-mentioned criteria can also be used here. In particular, the completeness criterion for boundary-element basis functions is satisfied if the interpolation function φ_{in} ($i = 1, \ldots, m$) of the form (2.2) are represented in the form of a complete polynomial, as a minimum, of the first degree in the variable $\eta \in \Delta s_n$ while the consistency criterion of the boundary elements is satisfied while writing the BEM equations because of equality of the nodal values of the desired solution at common nodes of contiguous elements (see Sect.5).

The realization of the proposed BEM variational formulation is considered in Sect.6 using the example of the algorithm to solve a unilateral variational problem for the BF (1.2).

4. We consider the BEM formulation on the basis of a variational problem for the GTF (1.4). The appropriate bilinear functional has the form

$$\Phi(\mathbf{u}, \mathbf{v}) = 2\int_G W(\mathbf{u}, \mathbf{v})\, dG + \frac{1}{\alpha}\int_S \mathbf{t}^{(\nu)}(\mathbf{u})\, \mathbf{t}^{(\nu)}(\mathbf{v})\, ds - \int_S \mathbf{t}^{(\nu)}(\mathbf{u})\, \mathbf{v}\, ds - \int_S \mathbf{u}\mathbf{t}^{(\nu)}(\mathbf{v})\, ds \tag{4.1}$$

The functional $\Phi(\mathbf{u})$ is minimized (see Sect.1) by solutions of the equation $\mathbf{A}\mathbf{u} = \mathbf{K}$ in the domain G which it is natural to consider as a constraint on the problem of finding $\min \Phi(\mathbf{u})$. This problem is single-valuedly solvable /8/ under appropriate conditions of single-valued solvability of the second boundary value problem of linear elasticity theory. We will consider the case of a homogeneous isotropic elastic medium. To solve the problem of finding $\min \Phi(u)$ we use the boundary-element approximation from Sects.2 and 3 in the form of the boundary (2.1), the domain $G_\Delta \subset G$ and the approximations

$$\mathbf{u}_N(x) = \boldsymbol{\delta}(x) + \sum_{n=1}^{N}\sum_{i=1}^{m}\sum_{k=1}^{K} U_{ik}\beta_{nk}(x), \quad x \in G_\Delta \tag{4.2}$$

$$\boldsymbol{\delta}(x) = \int_{G_\Delta} \boldsymbol{\Gamma}(x, y)\, \mathbf{K}(y)\, dG_\Delta(y), \quad y \in G_\Delta$$

Here U_{ik} are components of the desired nodal displacements of the boundary element Δs_n, the functions β_{nk} are determined by (3.2), $\boldsymbol{\delta}(x)$ is the bulk potential of a homogeneous isotropic elastic medium, and $\boldsymbol{\Gamma}(x, y)$ is the matrix of fundamental solutions of the statics equations of linear elasticity theory /9/.

The equality resulting from the material in Sect.2

$$\sum_{k=1}^{K} U_{ik}\beta_{nk} = \beta_{in}, \quad i = 1, \ldots, m, \quad n = 1, \ldots, N$$

is used later.

The bulk potential $\boldsymbol{\delta}(x)$ is a regular function /9/ (for $x \neq y$) and for any $x \in G_\Delta$ satisfies the equation $\mathbf{A}\boldsymbol{\delta} = \mathbf{K}(x)$; the functions $\beta_{in}(x)$ ($x \in G_\Delta$) satisfy the equation $\mathbf{A}\beta_{in} = 0$ (see (2.5)). Then for each N the approximations (4.2) satisfy the constraint $\mathbf{A}\mathbf{u}_N = \mathbf{K}$ almost everywhere in the domain G_Δ for the variational problem for the functional $\Phi_{G_\Delta}(\mathbf{u}_N)$, approximating the functional $\Phi(\mathbf{u})$.

Relationships that follow from the above and the Betti formulas /8/

$$2\int_{G_\Delta} W(\boldsymbol{\delta}, \mathbf{v})\, dG_\Delta = \int_{S_\Delta} \mathbf{t}^{(\nu_n)}(\boldsymbol{\delta})\, \mathbf{v}\, ds(y) + \int_{G_\Delta} \mathbf{K}\mathbf{v}\, dG_\Delta \tag{4.3}$$

$$2\int_{G_\Delta} W\Big(\sum_{i=1}^{m} \beta_{iN}, \mathbf{v}\Big)\, dG_\Delta = \int_{S_\Delta} \mathbf{t}^{(\nu_n)}\Big(\sum_{i=1}^{m} \beta_{iN}\Big)\, \mathbf{v}\, ds(y), \quad \beta_{iN} = \sum_{n=1}^{N} \beta_{in}.$$

and are valid for vector-functions $\mathbf{v}$ sufficiently smooth in $G_\Delta + S_\Delta$, will be needed later.

From the condition for a minimum of the functional $\Phi_{G_\Delta}(\mathbf{u}_N)$ we obtain a Ritz system to determine the coefficient U_{ik} which is written in general form thus

$$\Phi_{G_\Delta}\Big(\sum_{n=1}^{N}\sum_{i=1}^{m}\sum_{k=1}^{K} U_{ik}\beta_{nk} + \boldsymbol{\delta}, \beta_{rp}\Big) = 0, \quad r = 1, \ldots, N, \quad p = 1, \ldots, K \tag{4.4}$$

Using the expression of the bilinear functional approximating the functional (4.1) by the approximations (4.2), and the relationships (4.3) to eliminate volume integrals of the form $2\int_{G_\Delta} W(\mathbf{u}, \mathbf{v})\, dG_\Delta$ from (4.4) and also using relationships (3.4) and (3.5), we reduce system

(4.4) to a system of BEM equations in the desired nodal values U_{ik} $(i = 1, \ldots, m)$

$$\sum_{n=1}^{N}\Big\{\sum_{k=1}^{K} U_{ik}\Big[\frac{1}{\alpha}\int_{\Delta s_n(\eta)} c(\lambda,\mu)\frac{\partial\psi_k}{\partial \mathbf{v}_n} c(\lambda,\mu)\frac{\partial\psi_p}{\partial \mathbf{v}_n}|J|\,ds_n(\eta) - \tag{4.5}$$

$$\int_{\Delta s_n(\eta)} \psi_k c(\lambda,\mu)\frac{\partial\psi_p}{\partial \mathbf{v}_n}|J|\,ds_n(\eta)\Big]\Big\} =$$

$$-\sum_{n=1}^{N}\Big\{\sum_{p=1}^{K}\Big[\frac{1}{\alpha}\int_{\Delta s_n(\eta)} \mathbf{t}^{(\mathbf{v}_n)}(\delta)\, c(\lambda,\mu)\frac{\partial\psi_p}{\partial \mathbf{v}_n}|J|\,ds_n(\eta) -$$

$$\int_{\Delta s_n(\eta)} \delta c(\lambda,\mu)\frac{\partial\psi_p}{\partial \mathbf{v}_n}|J|\,ds_n(\eta)\Big]\Big\} + \int_{G_\Delta} \mathbf{K}\beta_{NK}\,dG_\Delta$$

$$\beta_{NK} = \sum_{r=1}^{N}\sum_{p=1}^{K}\beta_{rp}$$

where the functions $\beta_{rp}(x)$ are determined from (3.2).

The single-valued solvability of system (4.5) (under conditions for solvability of the corresponding boundary value problem in the domain G_Δ with boundary S_Δ) is set up, firstly, on the basis of symmetry of the system matrix, which follows from the symmetry of the bilinear functional $\Phi_{G_\Delta}(\mathbf{u}_N, \mathbf{v}_N)$ and the second relationship from (4.3), secondly, on the basis of positive-definiteness of the matrix, as follows from the positive-definiteness of the functional $\Phi_{G_\Delta}(\mathbf{u}_N)$ for the vectors $\mathbf{u}_N$ of the form (4.2) that satisfy the single-valuedness condition (3.7).

Solvability of the Ritz system obtained for minimizing the GTF with boundary norms

$$\|\mathbf{u}\|_{W_2^{1/2}(S)},\quad \|\mathbf{t}^{(\mathbf{v})}(\mathbf{u})\|_{W_2^{-1/2}(S)}$$

(see Sect.1) is investigated in /12/; also investigated is the stability of the Ritz process and recommendations are given for solving systems of equations of the form (4.5) by an iteration method.

The foundation for the proposed BEM formulation, using the GTF of the second boundary value problem of linear elasticity theory, is based on the fact that the boundary-element approximations (4.2) of the solution of the variational problem for this functional are "Ritz" approximations (see Sect.3). The convergence of the "Ritz" approximations in GTF minimization boundary value problems of linear elasticity theory is proved in /4/.

5. We now turn to questions concerning the numerical realization of the proposed variational formulations of the BEM. We examine the case of approximating the domain boundary by isoparametric curvilinear boundary elements /2/. Then the geometric nodes for the approximation of the boundary and the functional nodes for the approximation of the solution are in agreement and the very same systems of basis BEM functions are used for the approximation. We will show that in this case the coefficients for the unknowns of the system of BEM equations of the form (3.6) are determined identically by the coefficients of a classical Ritz system /8/ for the BF minimization problem of the form (1.1) with the constraint $\mathbf{A}\varphi = 0$ in G.

For simplicity we consider the two-dimensional case corresponding to the plane problem of elasticity theory. Then to approximate the boundary we use one-dimensional isoparametric curvilinear boundary elements with a quadratic change in the BEM basis elements at points of the element; such an element has three nodes. We perform certain auxiliary constructions that are associated with elements of the differential geometry of surfaces, particularly with the properties of the Jacobians of mutually inverse differential mappings /13/. In the case of one-dimensional boundary elements Δs_n the mapping $\Delta s_n(y) \to \Delta s_n(\eta)$ is characterized by the Jacobian

$$|J| = \{(\partial_\eta y_1)^2 + (\partial_\eta y_2)^2\}^{1/2},\quad \partial_\eta y_i \equiv \frac{\partial y_i}{\partial \eta},\quad i = 1, 2 \tag{5.1}$$

We recall (see Sect.2) that $y = (y_1, y_2)$ are Cartesian coordinates, while η is the local coordinate of points belonging to the boundary element Δs_n.

In the case of the isoparametric element $\Delta s_n(\eta)$ the relation between the Cartesian coordinates (y_1, y_2) and the local coordinate η is the following:

$$y_i = \sum_{k=1}^{K} y_{ik}\psi_k(\eta),\quad i = 1, 2,\quad \eta \in \Delta s_n,\quad n = 1, \ldots, N \tag{5.2}$$

where (y_{1k}, y_{2k}) are Cartesian coordinates of the node k of the element Δs_n, while $\psi_k(\eta)$ are the second-order BEM basis functions in η and $K = 3$ is the number of interpolation nodes within each element Δs_n. At the points of the approximating boundary (2.1) with the external normal $\mathbf{v}_n(y_1, y_2)$ the normal derivative of the basis functions $\psi_k(y_1, y_2)$ is defined as

$$\partial_{\mathbf{v}_n}\psi_k = \partial_{y_1}\psi_k \cos(\mathbf{v}_n, y_1) + \partial_{y_2}\psi_k \cos(\mathbf{v}_n, y_2), \quad \partial_{\mathbf{v}_n} \equiv \frac{\partial}{\partial \mathbf{v}_n} \tag{5.3}$$

Furthermore, we use the expression (/13/, p.87) for the direction cosines of the normal $\mathbf{v}_n$ at points of the element Δs_n whose geometry is approximated by using the basis functions $\psi_k(\eta)$ by the relationships (5.2)

$$\cos(\mathbf{v}_n, y_i) = \partial_{y_i}\psi_k \{(\partial_{y_1}\psi_k)^2 + (\partial_{y_2}\psi_k)^2\}^{-1/2}, \quad i = 1, 2$$

Substituting these relationships into (5.3) and taking account of the equality $\partial_{y_i}\psi_k = \partial_\eta\psi_k\partial_{y_i}\eta$ $(i = 1, 2)$, we obtain

$$\partial_{\mathbf{v}_n}\psi_k = \partial_\eta\psi_k \{(\partial_{y_1}\eta)^2 + (\partial_{y_2}\eta)^2\}^{1/2} = \partial_\eta\psi_k \,|J|^{-1}$$

where $|J|^{-1}$ is the Jacobian of the inverse mapping $\Delta s_n(\eta) \to \Delta s_n(y)$ (see (5.1)).

The Jacobians of mutually inverse continuously differentiable mappings are related by the relationship $|J|^{-1}|J| = 1$ (/13/, p.62). Then the coefficients for the unknowns in the system (3.6) are transformed as follows

$$\int\limits_{\Delta s_n(\eta)} \partial_{\mathbf{v}_n}\psi_k\psi_p |J|\, d\eta = \int\limits_{\Delta s_n(\eta)} \partial_\eta\psi_k\psi_p\, d\eta, \quad k, p = 1, \ldots, K$$

Let us present a method for compiling the equations in system (3.6). Since each of the basis functions ψ_k of the n-th element is identically zero at contiguous points of the $(n-1)$-th and $(n+1)$-th elements, each of the equations of the system will contain components for $n = z$, and the remaining components vanish. There are two kinds of equations. For the middle node of the n-th element the left-hand side of the equation is written in the form

$$-\Phi_{i1}I_1^- + \Phi_{i2}(I_2^- + I_2^+) - \Phi_{i3}I_3^+ \tag{5.4}$$

$$I_k^\pm = \int\limits_0^{\pm 1} \partial_\eta\psi_k\psi_k\, d\eta; \quad k = 1, 2, 3; \quad i = 1, 2$$

where Φ_{ik} are the desired nodal values, and η is a dimensionless coordinate that varies within the limits of the element between -1 and $+1$. For the common node of the adjacent n-th and $(n+1)$-th elements, the left side of the equation will have the form

$$\Phi_{i2}I_2^+ - \Phi_{i1}^{(n+1)}(I_1^{(n+1)-} + I_3^+) + \Phi_{i2}^{(n+1)}I_2^{(n+1)-} \tag{5.5}$$

$$I_k^{(n+1)-} = \int\limits_0^{-1} \partial_\eta\psi_k^{(n+1)}\psi_k^{(n+1)}\, d\eta; \quad k, i = 1, 2$$

Here $\Phi_{i1}^{(n+1)} = \Phi_{i3}^{(n)}$, i.e. the desired nodal value for the first node of the $(n+1)$-th element agrees with the nodal value for the third node of the n-th element (numbering of the nodes is counter-clockwise). Therefore, at points of the approximating boundary S_Δ the solution will be continuous, and therefore, the consistency condition for the boundary elements Δs_n is satisfied (see Sect.3).

6. The unilateral variational problem for the BF (1.2) in the set of vector-functions $\boldsymbol{\varphi} \in V(S_1)$ under the linear constraints (1.3) is equivalent /5/ to the unilateral boundary value problem

$$\mathbf{A}\boldsymbol{\varphi} = 0 \ \text{in} \ G, \quad \varphi^{(\nu)}|_{S_1} \geqslant 0, \quad [\mathbf{t}^{(\nu)}(\boldsymbol{\varphi}) + \mathbf{t}^{(\nu)}(\mathbf{u})]_{S_1} \geqslant 0 \tag{6.1}$$

$$(\varphi^{(\nu)}[\mathbf{t}^{(\nu)}(\boldsymbol{\varphi}) + \mathbf{t}^{(\nu)}(\mathbf{u})]_{S_1} = 0), \quad \mathbf{t}^{(\nu)}(\boldsymbol{\varphi})|_{S_2} = 0$$

where $\mathbf{t}^{(\nu)}(\mathbf{u})$ is a given normal stress vector at points of the boundary of possible contact S_1 (here the vector $\mathbf{t}^{(\nu)}(\mathbf{u})$ is not related to the displacement vector $\mathbf{u}^*$ (see Sect.1)). Problem (6.1) is a problem of the generalized Signorini problem type that is solvable /7/ under the condition $\mathbf{t}^{(\nu)'}(\mathbf{u})|_{S_1} < 0$.

Numerical realization of the proposed BEM formulation on the basis of a variational problem for a BF is examined in an example of a classical plane contact problem of insertion (without taking friction into account) of an absolutely rigid stamp into an elastic isotropic half-plane which can be formulated as problem (6.1). A duality algorithm (of the Udzawa-Arrow-Gurwitz type /14/), based on the reciprocal formulation of the Lagrangian maximum problem, is used to solve this problem as a variational problem for a BF of the form (1.2) with linear and convex constraints on the allowable functions. The Lagrange multipliers

$\{\lambda_j\}_{j=0,1,2,\ldots}$ were used to remove the convexity constraint $\varphi \in V(S_1)$ and the mechanical interpretation of these multipliers is: λ_j is the intensity of the distributed normal support reaction at points of the set $S_1^\circ \subset S_1$ (unknown a priori) at which the stamp contact with the boundary S_1 exists; here $\varphi_0^{(\nu)}|_{S_1^\circ} = 0$, where $\varphi_0 \in V(S_1)$ is a solution of problem (6.1).

When solving the duality problem for a minimum (with fixed Lagrange multipliers), a Ritz process is used on the basis functions in the form of the potentials (2.4) satisfying the linear constraint of the problem, the equilibrium equation. Here the boundary S_1 of possible contact was approximated by isoparametric second-order boundary elements and the solution of the boundary-element approximation of the problem on the minimum in the BEM basis functions

$$\psi_1 = {}^1/_2\eta(1-\eta),\quad \psi_2 = (1-\eta)(1+\eta),\quad \psi_3 = {}^1/_2\eta(1+\eta)$$

was realized according to the material in Sects.2, 3 and 5. When writing the boundary-element equations according to (5.4) and (5.5) at the extreme nodes of the discretized contact boundary $S_{1\Delta}$, the nodal values of the stress vector $\mathbf{t}^{(\nu)}(\varphi)$ were assumed to be zero.

The maximum problem under convexity constraints on the Lagrange multipliers was solved by the gradient descent method (generalization of the Frank-Wolf method /14/). The condition for halting the iteration process in λ_j was given as

$$\sum_{n=1}^{N} \Big| \int_{\Delta s_n} \varphi_{\lambda_j n} [\mathbf{t}^{(\nu_n)}(\varphi_{\lambda_j n}) - \mathbf{p}]\, ds_n \Big| < \varepsilon \tag{6.2}$$

where ε is a given positive number governing the required accuracy of the iteration process in the number of iterations j for a fixed number N of boundary elements Δs_n of the boundary $S_{1\Delta}$. The vector $\mathbf{t}^{(\nu)}(\mathbf{u})$ (see (6.1)) was given as: $\mathbf{t}^{(\nu)}(\mathbf{u}) = -\mathbf{p}$, where $\mathbf{p}(y) > 0$ $(y \in S_1)$ is the contact normal pressure function under a stamp with a definite stamp-surface geometry in the contact zone under the action of a force

$$P = \int_{-a}^{a} \mathbf{p}(y)\, dy$$

on the stamp, where a is the halfwidth of the symmetric possible contact zone relative to the stamp axis /15/. The function $\mathbf{p}(y)$ for a circular stamp bounded by the second-order curve $f(y) = y^2/(2R)$ within the limits of the possible contact zone S_1 was taken from /15/ (p.65). Two versions of the partition of the contact boundary S_1 into boundary elements were considered for the greatest depth taken (along the stamp axis of symmetry) for the stamp insertion $h = 0.02R$ and the corresponding halfwidth $a = 0.2R$ of the possible contact zone. For six elements and $\varepsilon = 5\times10^{-2}$ under conditions (6.2), the greatest error (at the point $y = 0$ on the stamp axis of symmetry) in the values of $\mathbf{p}$ and $\mathbf{t}^{(\nu)}(\varphi_{\lambda_j N})$ was $\delta \approx 16\%$ $(j = 14)$. For twelve elements the following error values are obtained: $\delta \approx 14.5\%$ $(j = 18)$ for $\varepsilon = 5\times10^{-2}$; $\delta \approx 8\%$ $(j = 29)$ for $\varepsilon = 10^{-2}$, and $\delta \approx 1.5\%$ $(j = 55)$ for $\varepsilon = 10^{-3}$ (computations performed on an ES-1022 computer). It was established that an increase in the number of iterations j affects the diminution in the error δ to a greater degree than an increase in the number of boundary elements N.

In substance, the example examined is corroborating the duality algorithm used, in the sense that the given contact stresses $\mathbf{p}(y)$, obtained as a result of solving the plane contact problem by the method of complex function theory /15/, are compared with the contact stresses $\mathbf{t}^{(\nu)}(\varphi_{\lambda_j N})$, obtained as a result of solving the unilateral variational problem for the BF (1.2).

7. We will make certain deductions. Unlike the BEM formulations recalled at the beginning of the paper, the systems of BEM equations have symmetric matrices in the proposed variational formulations. Composed with the BEM formulation utilizing boundary integral equations, there is no singularity in calculating the surface integrals in the stage of system matrix formation. Values of the generalized Trefftz functional of the problem being solved, calculated by the approximate solutions /4/, or values of the functional of the dual variational problem /3/ can be used for an a posteriori estimate of the error in the boundary-element approximations, (in the BEM formulation using boundary functionals).

At the same time the possibilities of practical utilization of the proposed BEM formulations are constrained by the boundary value problems for which Green's functions exist. If it is taken into account that Green's function is required in explicit form to obtain the solution at points of the domain by the boundary values found, then application of the BEM formulation is possible in contact problems of linear elasticity theory in which we are interested in the displacement and stress distribution at points of the contact boundary. It is sufficient for such problems to satisfy the conditions on the data of the problem for which Green's function exists; its construction in explicit form, however, is not necessary.

The possibility should be noted of a dual formulation of BEM on the basis of the dual variational problem for boundary functionals in terms of the surface stresses /3/.

In conclusion, we note that the variational BEM formulations elucidated for linear elasticity theory problems can also be used for other elliptic boundary value problems by using the variational construction of problems for boundary functionals resulting from orthogonal expansions on the boundary of a domain in elliptic boundary value problems /16/.

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Translated by M.D.F.